MATHEMATICS

Mob. : 9470844028
9546359990

AIM POINT MATHEMATICS

DIR. FIROZ AHMAD
M.Sc. (Maths), B.Ed, M.Phil (Maths)

RAM RAJYA MORE, SIWAN

XIth, XIIth, TARGET IIT-JEE (MAIN + ADVANCE) & COMPATETIVE EXAM FOR XII (PQRS)

CONTINUITY AND DIFFERENTIABILITY

& Their Properties

CONTENTS

THINGS TO REMEMBER

✷ Continuity

The geometrical significance of continuity is that if the function is continuous, its graph does not bear a break otherwise it is discountinuous. The point where the graph of the function breaks is called the point of discontinuity. Thus, a function is continuous at a point if limit exist at this point and equal to the value of the function at this point.

Continuty at a Point

A function $f(x)$ is said to be continuous at point x = a in its domain, if $\lim_{h\to 0} f(a + h) = \lim_{h\to 0} f(a - h) = f(a)$

ie, left hand limit, right hand limit and value of function at point x = a exists and are equal.

Continuity in an open Interval

A function $f(x)$ is said to be continuous in the open interval]a, b[or $a < x < b$, if it is continous at each point of the interval.

Continuity from Left and from Right

Let $f(x)$ be a function defined o and open interval I and let $a \in I$ then f is continuous from the left at a if $\lim_{x\to a^-} f(x)$ exists and is equal to $f(a)$. Similarly, $f(x)$ is said to be continuous from te right at a, if $\lim_{x\to a^+} f(x)$ exists and is enqual to $f(a)$.

Continuity in a Closed Interval

Let $f(x)$ be a function defined on the closed interval [a, b]. Then , $f(x)$ is said to be continuous on the closed interval [a,b], if it is.

(a) Continuous from the right at a

(b) Continuous from the left at b

(c) Continuous on the open interval]a, b[.

✷ Discontinuous Function

If $f(x)$ is not continuous at x = a, then $f(x)$ is said to be discontinuous at x = a and this point is called a point of discontinuity.

✷ Different Kinds of Discontinuity :

1. Removable Discontinuity

A function $f(x)$ is said to have a removable discontinuity at a point α if $f(\alpha + 0)$, $f(\alpha - 0)$ and $f(\alpha)$ exist but $f(\alpha + 0) = f(\alpha - 0) \neq f(\alpha)$, The function can be made continuous by defining it in such a way that $\lim_{x\to\alpha} f(x) = f(\alpha)$.

eg, $f(x) = [\sin x]$ where $x \in (0, \pi)$ has a removable discontinuity at $x = \frac{\pi}{2}$

$$\lim_{x\to\frac{\pi}{2}} [\sin x] = 0 \text{ but } f\left(\frac{\pi}{2}\right) = 1$$

To remove this we redefine $f(x)$ as follows

$$f(x) = \begin{cases} [\sin x], & x \in \left(0, \frac{\pi}{2}\right)\left(\frac{\pi}{2}, \pi\right) \\ 0, & x = \frac{\pi}{2} \end{cases}$$

Now, $f(x)$ is continuous for $x \in (0, \pi)$.

2. Discontinuity of First Kind

A function $f(x)$ is said to have a discontinuity of the first kind or ordinary discontinuity at 'α' if $f(\alpha + 0)$ and $f(\alpha + 0)$ both exist but are not equal.

3. Discontinuity of Second Kind

A function $f(x)$ is said to have a discontinuity of the second kind at 'α' if none of the limits $f(\alpha + 0)$ and $f(\alpha + 0)$ exist.

4. Infinite Discontinuity

If either or both of $f(\alpha + 0)$ and $f(\alpha - 0)$ be infinite ($+\infty$ or $-\infty$), then $f(x)$ has infinite discontinuity at $x = \alpha$.

✷ **Properties of Continuous Function :**

1. If y = $f(x)$ and y = g(x) are continuous functions at x = a, then functions $f(x) \frac{+}{\times} g(x)$ are also continuous at x = a, only in case of $f(x)$ g(x), g(a) $\neq$ 0.
2. If y = $f(x)$ is continuous at x = a and y = g(x) is discontinuous at x = a, then $f(x) \times g(x)$ may be continuous function eg, $f(x) = x$ and $g(x) = [x]$, $f(x)$ is continuous at $x = 0$ and g(x) is discontinuous at $x = 0$ and $x\,[x]$ is continuous at $x = 0$.
3. If y = $f(x)$ and y = g(x) are continuous functions at x = a, then $f(x) \overset{+}{\underset{\div}{\times}} g(x)$ may be continuous function at x = a, eg, $f(x) = [x]$, $g(x) = \{x\}$, both functions are discontinuous at integers but $f(x) + g(x) = x$ continuous everywhere and let $f(x) = [x]$, $g(x) = \frac{x}{[x]}$, bothe functions are discontinuous at $x = 2$ but $f(x).g(x) = x$, continuous at $x = 2$.
4. Trigonometric function, inverse trigonometric function, logarithmic function, exponentail function and modulus function are continuous in their domain.
5. Every polynomial is continuous at every point of the real line.
6. Every rational function i continuous at every point where its denominator is not zero.

✷ Differentiability :

Let $f(x)$ be a real valued function defined on an open interval (a,b) where c e (a, b). Then, $f(x)$ is said to be differentiable of derivable at x = c if $\lim\limits_{x\to c}\dfrac{f(x)-f(c)}{x-c}$ exists finitely.

This limit is called the derivative of differential coefficient of the function $f(x)$ at x = c and is denoted by $f\,'$(c) of Df(c) or $\dfrac{d}{dx}\left(f(x)\right)_{x=c}$.

Thus, $f(x)$ is differentiable at x = c.

$\Leftrightarrow \lim\limits_{x\to c}\dfrac{f(x)-f(c)}{x-c}$ exists finitely.

Right Hand Derivative

Let $f(x)$ be a function and 'a' be a point in the domian of f. Then, right hand derivative is denoted by R$f\,'$(a) abd defubed as RHD = R$f\,'$(a) $\lim\limits_{x\to a^+}\dfrac{f(x)-f(a)}{x-a}=\lim\limits_{h\to 0}\dfrac{f(a+h)-f(a)}{h}$.

Left Hand Derivative

Let $f(x)$ be a function and 'a' be a point in the domian of f. Then, right hand derivative is denoted by L$f\,'$(a) abd defubed as RHD = L$f\,'$(a) $\lim\limits_{x\to a^-}\dfrac{f(x)-f(a)}{x-a}=\lim\limits_{h\to 0}\dfrac{f(a-h)-f(a)}{-h}$.

If L$f\,'$(a) R$f\,'$(a), then function $f(x)$ is not differentiable at x = a.

Differentiability is an Inteval

A function y = $f(x)$ defined on an open interval (a, b) is said to be differentiable in an open interval (a,b), if it is differentiable at each point of (a, b).

A function y = $f(x)$ defined on a closed interval [a, b] is said to be differentiable in closed interval [a,b], if it is differentiable at each point of an open interval (a, b) and $\lim\limits_{x\to a^+}\dfrac{f(x)-f(a)}{(x-a)}$ and both $\lim\limits_{x\to b^-}\dfrac{f(x)-f(b)}{(x-b)}$ exist.

OR

A function f is said to be a differentiable function, if it is differentiable at every point of its domain.

✷ Relation between Continuity and Differentiablility :

If a functionis differentiable at a point, then it is necessarily continuous at that point. But the converse is not necessarily true, ie, every continuous function need not be differentiable.

✷ Properties of differentiability :

1. Every polynomial function, exponential function and constant function are differentiable at each $x \in$ R.
2. Logarithmic functions, Trigonometirc and inverse trigonometric functions are differentiable in their respective domains.
3. The sum, difference, produch and quotient of two differentiable function is differentiable.
4. The composition of differentiable function is a differentiable function.

Note :

- A composite function $fog(x)$ is continuous at $x = \text{a}$, if g is continuous at $x = \text{a}$ and f is continuous at g(a).
- If the graph of a function have a sharp edge at any point, then function is not differentiable at that point.